# Beyond quasi-linear utility: strategy/false-name-proof multi-unit auction protocols

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## Abstract

*We develop strategy/false-name-proof multi-unit auction protocols for non-quasi-linear utilities. One almost universal assumption in auction theory literature is that each bidder has quasi-linear utility. However, in practice, a bidder might have some kind of financial condition including budget constraints. The Vickrey-Clarke-Groves (VCG) protocol is designed to be truthful under the quasi-linear assumption and will break down if this assumption does not hold. We show with a simple modification, the VCG can handle non-quasi-linear utilities. However, there are possibilities that the modified VCG sacrifices significant efficiency loss, since it only uses the gross utilities for determining tentative allocation and payments. Also, it has been shown that the VCG is vulnerable to a false-name bid which is a new type of cheating on the Internet. To improve efficiency without collecting the entire utility functions and guarantee false-name-proofness, we develop a false-name-proof open ascending auction protocol.*

## 1   Introduction

Internet auctions have become an important part of Electronic Commerce, and the amount of transactions is increasing annually. Various goods are traded in Internet auctions.

In recent years, many theoretical/practical studies on Internet auctions have been conducted [5, 10]. One almost universal assumption of these works is that each bidder has quasi-linear utility, i.e., a bidder's utility is defined as the difference between her gross utility of allocated goods and her payment. One notable exception is works on budget-constrained bidders in single-item/multi-unit auctions [2, 3, 6]. In these works, a bidder is assumed to have budget constraints, i.e., she cannot pay more than a

predefined budget limit. Her utility becomes minus infinity when her payment exceeds her limit.

We can generalize such utilities to more general cases, such as piece-wise-quasi-inear utility. For example, a bidder can pay up to certain amount from her own budget. If the payment exceeds that amount, she needs a loan and must pay interest. Such a utility can be represented as a separable utility function, where a bidder's utility is defined as the difference between her gross utility and an increasing function of her payment. In more general cases, we consider non-separable utility in which the allocated goods and payment interact with each other and determine the utility.

For single-item/multi-unit auctions, we can apply the well-known Vickrey-Clarke-Groves (VCG) protocol [4, 8, 13] when bidders have quasi-linear utilities. However, the VCG critically depends on the quasi-linear assumption due to Gibbard-Satterthwaite impossibility theorem [7, 12] and will break down if this assumption does not hold. For example, in [2], Borgs *et al.* showed that the VCG is no longer strategy-proof if bidders have budget limits.

Also, for Internet auctions, Yokoo *et al.* pointed out the possibility of a new type of fraud called false-name bids that exploit the anonymity available on the Internet [11, 16]. False-name bids are submitted under fictitious names, e.g., multiple e-mail addresses. Such deception is very difficult to detect, since identifying each participant on the Internet is virtually impossible. Although several false-name-proof multi-unit auction protocols have been developed, they also assume quasi-linear utilities.

We show a simple modification in which the VCG can handle non-quasi-linear utilities. The basic idea of this modification is that tentative allocation and payments are determined assuming quasi-linear utilities, but each bidder can choose the actual number of units to obtain based on her non-quasi-linear utility. More specifically, we utilize a general framework for describing strategy-proof protocols called the Price-Oriented Rationing-Free (PORF) protocol

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introduced in [14]. We prove that if any auction protocol, which can be described as the PORF protocol, satisfies allocation feasibility in quasi-linear cases, it is guaranteed to satisfy allocation feasibility in non-quasi-linear utility case as long as some additional conditions are satisfied. The VCG can be described as the PORF protocol and satisfy allocation feasibility. Moreover, a new false-name-proof multi-unit auction protocol based on an existing false-name-proof protocol, which is called Groves Mechanism with SubModular Approximation (GM-SMA)[15], can also handle non-quasi-linear utilities with the same modification. These modified protocols only use the gross utility of each bidder. Requiring only gross utility can be an advantage since collecting entire utility functions can be costly. However, determining tentative allocation and payments without considering actual non-quasi-linear utilities can cause significant efficiency loss.

To improve efficiency without collecting entire utility functions, we develop a new false-name-proof, open ascending auction protocol called the Non-Quasi-linear Ascending-price OPtion allocation (NQ-AOP) protocol. The NQ-AOP is based on an existing false-name-proof protocol called Ascending-price OPtion allocation protocol (AOP) [9] which is developed by modifying the Ausubel auction [1]. In the NQ-AOP, a bidder declares a demand for a series of prices announced by the auctioneer. For each announced price and for each bidder, the auctioneer calculates the aggregated demand of other bidders and allocates an option to the bidder to obtain the remaining units at the current price. When the total demand becomes less than the available units, each bidder can choose the optimal option among the allocated options. Our simulation results show that this protocol obtains better social surplus than the modified protocols when bidders have budget constraints.

## 2 Model

Assume $K$ units of homogeneous goods and a set of bidders $N = \{1, 2, \ldots, n\}$ where $n \geq K$. Bidder $i$ determines her utility by privately observing a parameter or signal, $\theta_i$. We refer to $\theta_i$ as the *type* of bidder $i$ and assume $\theta_i$ is drawn from set $\Theta$. Let $u(\theta_i, k, p)$ denote the utility of the bidder with type $\theta_i$ when she obtains $k$ units and pays price $p$. We assume $u(\theta_i, 0, 0)$ is normalized to 0.

First, we define quasi-linear utility which is commonly used in auction theory literature. $u(\theta_i, k, p)$ is a *quasi-linear utility* if it is defined as the difference between the gross utility of the allocated goods and the payment:

$$u(\theta_i, k, p) = v(\theta_i, k) - p. \tag{1}$$

Here, $v(\theta_i, k)$ indicates bidder $i$'s valuation of $k$ units and is identical to $u(\theta_i, k, 0)$.
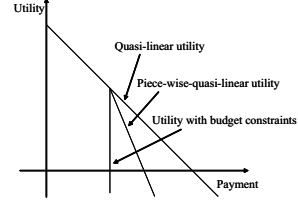


**Figure 1. Non-quasi-linear utilities**

Next, *separable utility* $u(\theta_i, k, p)$ is represented as the difference between gross utility and an increasing function of payment:

$$u(\theta_i, k, p) = v(\theta_i, k) - f(\theta_i, p). \tag{2}$$

Here, we assume that $f$ is an increasing function of $p$ and is normalized at $f(\theta_i, 0) = 0$. Without loss of generality, we assume $\forall p, f(\theta_i, p) \geq p$ holds.

Utility with budget constraints and a piece-wise quasi-linear utility can be represented as special case of a separable utility function. Separable utility is a *utility with budget constraints*, if for a budget limit $b_i$,

$$f(\theta_i, p) = \begin{cases} p, & p \leq b_i \\ \infty, & \text{otherwise.} \end{cases} \tag{3}$$

We can generalize utility with budget constraints to more general cases, such as piece-wise quasi-linear utility. A separable utility is *piece-wise quasi-linear* if, for a series of budget limits $b_{i,1}, b_{i,2}, \ldots, b_{i,t}$,

$$f(\theta_i, p) = \begin{cases} \alpha_1 p, & p \leq b_{i,1} \\ \alpha_j p, & b_{i,j-1} \leq p < b_{i,j} \\ \alpha_t p, & b_{i,t} < p. \end{cases} \tag{4}$$

Here, we assume $\alpha_j \geq 1$ for all $j$. With piece-wise quasi-linear utility, we can represent a case where a bidder can pay up to certain amount from her own budget, but if her payment exceeds that amount, she needs a loan and must pay interest. Figure 1 illustrates a quasi-linear utility, a utility with budget constraints, and a piece-wise quasi-linear utility.

In the most general case, we can consider *inseparable utility* in which the allocated goods and the payment can interact with each other and determine the utility. For inseparable utility $u(\theta, k, p)$, we assume the following condition holds:

$$\forall k, \ \forall p \leq p' \quad u(\theta_i, k, p) - u(\theta_i, k, p') \geq p' - p. \tag{5}$$

This condition means that the utility decrease caused by the payment increase is at least (or more than) linear.

Now, we introduce several properties that auction protocols should satisfy. An auction protocol is *individually rational* if each auction participant does not suffer any loss.

An auction protocol is *strategy-proof* if for each bidder, declaring her true type is a dominant strategy, i.e., the optimal strategy for maximizing her utility regardless of other bidders' actions. An auction protocol is *false-name-proof* if for each bidder, declaring her true type using a single identifier, while she can use false-name bids, is a dominant strategy.

## 3  Limitation of VCG

The VCG is recognized as a strategy-proof protocol in quasi-linear utility case. The VCG for $K$ units of homogeneous goods is defined as follows.

- Bidder $i$ declares her type $\theta_i$ to the auctioneer. Let us denote $(v(\theta_i, 1), \cdots, v(\theta_i, k), \cdots, v(\theta_i, K))$ as a valuation vector based on declared type.

- The auctioneer determines an allocation and payments. For a set of feasible allocation $\mathbf{K}$ where $\mathbf{K} = \{\mathbf{k} = (k_1, \cdots, k_N) | \sum_{i \in N} k_i \leq K\}$, we define $\mathbf{k}^*$ as an optimal allocation that maximizes the sum of the declared valuations in $\mathbf{K}$, formally, $\mathbf{k}^* = \arg\max_{\mathbf{k}} \sum_{i \in N} v(\theta_i, k_i)$. Aggregated utility function $V^*$ is defined as follows: $V^*(K, \Theta_N) = \sum_{i \in N} v(\theta_i, k_i^*)$ where $k_i^* \in \mathbf{k}^*$. Then bidder $i$'s price for $k_i^*$ units, $p_i(k_i^*)$, is calculated as follows:

$$p_i(k_i^*) = V^*(K, \Theta_{N \setminus \{i\}}) - V^*(K - k_i^*, \Theta_{N \setminus \{i\}}). \tag{6}$$

Borgs *et al.* [2] showed an example where the VCG does not satisfy strategy-proofness for bidders with budget constraints. In their example, bidder $i$ with budget limit $b_i$ is assumed to have linear gross utility, i.e., her utility for $k$ units is given as $kc_i$, where $c_i$ is a unit value. The VCG is applied to modified valuation vector $(v'(\theta_i, 1), \cdots, v'(\theta_i, K))$ where $v'(\theta_i, k) = \min(kc_i, b_i)$. Applying this modified valuation vector guarantees that each winner's payment does not exceed the amount of her budget.

**Example 1** *Assume 2 bidders take part in an auction for 2 units of homogeneous goods. Suppose $c_i$ and $b_i$ as $(c_1, b_1) = (10, 10)$ and $(c_2, b_2) = (3, 5)$.*

*We compute the VCG payment using modified valuation vector $(v'(\theta_i, 1), v'(\theta_i, 2)) = (\min(c_i, b_i), \min(2c_i, b_i))$. Bidders 1 and 2's valuation vectors are calculated as $(10, 10)$ and $(3, 5)$. Thus, bidders 1 and 2 can obtain 1 unit at prices 2 and 0, respectively. The utility of bidder 1 is calculated as $10 - 2 = 8$.*

*If bidder 1 declares her bids as $(c_1, b_1) = (5, 10)$ by underbidding a value per unit, she can get 2 units at price 5. As a result, bidder 1's utility becomes $20 - 5 = 15$. Bidder 1 can increase her utility by underbidding.*

## 4  New sealed-bid auction protocols

We show that the VCG can handle non-quasi-linear utilities with simple modification. Moreover, we introduce a false-name-proof auction protocol called multi-unit GM-SMA that can handle non-quasi-linear utilities.

### 4.1  PORF protocol

We describe a general framework for describing strategy-proof protocols called a PORF protocol. Describing a protocol as a PORF protocol simplifies proving that it is strategy/false-name-proof. The following definition of a PORF protocol covers both cases of quasi-linear and non-quasi-linear utilities.

- Bidder $i$ declares a type $\tilde{\theta}_i$, which is not necessarily her true type $\theta_i$.

- For bidder $i$ and for any number of unit $k \leq K$, each price is defined. This price must be determined independently of $i$'s declared type $\tilde{\theta}_i$, while it can be dependent on the declared types of other bidders.

- For bidder $i$, the optimal number of units to maximize $u(\theta_i, k, p_i(k))$ is allocated. Here, we refer $p_i(k)$ to bidder $i$'s price for $k$ units and $p_i(k) \leq p_i(k')$ holds for $\forall k < k'$. If multiple optimal allocations exist, one of them is determined.

Furthermore, a PORF protocol guarantees individual rationality, since each bidder can choose the optimal number of units to maximize her utility after annoucing her price for each unit.

Now, we show if a protocol described as a PORF protocol satisfies allocation feasibility for quasi-linear utilities, it can also satisfy allocation fesibility for non-quasi-linear utilities. We introduce one additional condition called *steepness condition*. Inseparable utility $u(\theta_i, k, p_i(k))$ satisfies steepness condition if the following condition holds: $\forall k < k', p_i(k)$,

$$
\begin{aligned}
&u(\theta_i, k, 0) - u(\theta_i, k, p_i(k)) \\
&\leq\ u(\theta_i, k', 0) - u(\theta_i, k', p_i(k)). \tag{7}
\end{aligned}
$$

This condition means that if the allocated number of units increases, the utility decrease caused by the price increase does not become smaller. For a separable utility function, this condition is automatically satisfied, since $u(\theta_i, k, 0) - u(\theta_i, k, p_i(k))$ equal $f(\theta_i, p_i(k))$ for all $k$.

**Theorem 1** *If a protocol described as a PORF protocol can satisfy allocation feasibility for quasi-linear utilities, in non-quasi-linear utility case with the steepness condition, it can also satisfy allocation feasibility when the auctioneer calculates a tentative allocation and the prices only based on the gross utility.*

**Proof 1** *Assume bidder $i$'s utility is maximized by obtaining $k$ units if we assume bidder $i$'s utility is quasi-linear. Then for all $k' > k$, the following condition must hold:*

$$u(\theta_i, k, 0) - p_i(k) \geq u(\theta_i, k', 0) - p_i(k'). \quad (8)$$

*We will prove $u(\theta_i, k, p_i(k)) \geq u(\theta_i, k', p_i(k'))$, i.e., bidder $i$ does not want to obtain $k'$ units, where $k' > k$. Thus, we can guarantee allocation feasibility even if the utilities are non-quasi-linear. From formulas (5), we obtain:*

$$u(\theta_i, k, p_i(k)) \geq u(\theta_i, k, 0) - u(\theta_i, k', 0) + u(\theta_i, k', p_i(k)).$$

*Also from (7), we obtain:*

$$u(\theta_i, k, 0) - u(\theta_i, k', 0) \geq p_i(k) - p_i(k').$$

*Also from (8), we get:*

$$u(\theta_i, k', p_i(k)) \geq p_i(k') - p_i(k) + u(\theta_i, k', p'_i(k')).$$

*Therefore,*

$$
\begin{aligned}
u(\theta_i, k, p_i(k)) &\geq u(\theta_i, k, 0) - u(\theta_i, k', 0) \\
&\quad + u(\theta_i, k', p_i(k)) \\
&\geq p_i(k) - p_i(k') + u(\theta_i, k', p_i(k)) \\
&\geq p_i(k) - p_i(k') + p_i(k') \\
&\quad - p_i(k) + u(\theta_i, k', p_i(k')) \\
&= u(\theta_i, k', p_i(k'))
\end{aligned}
$$

*Thus, we obtain $u(\theta_i, k, p_i(k)) \geq u(\theta_i, k', p_i(k'))$.*

## 4.2 Modified VCG

When each bidder's utility is quasi-linear, the VCG can be described as a PORF protocol. The VCG price of bidder $i$ for any $k$ units is determined as follows:

$$p_i(k) = V^*(K, \Theta_{N \setminus \{i\}}) - V^*(K - k, \Theta_{N \setminus \{i\}}). \quad (9)$$

When each bidder's utility is non-quasi-linear, we redefine an aggregated utility function $V^*$ as follows:

$$V^*(K, \Theta_N) = \max_{\mathbf{k} \in \mathbf{K}} \sum u(\theta_i, k_i, 0). \quad (10)$$

In the VCG for non-quasi-linear utilities, the auctioneer calculates a tentative allocation and the prices only based on the gross utility, i.e., applying formulas (9) and (10). Since the VCG satisfies allocation feasibility for quasi-linear utility [15], it is also guaranteed to satisfy allocation feasibility for non-quasi-linear utility by Theorem 1.

**Example 2** *Assume $2$ bidders in an auction for $2$ units, and bidders $1$ and $2$'s gross utility $(v(\theta_i, 1), v(\theta_i, 2))$ are $(10, 15)$ and $(8, 16)$, respectively.*

*The VCG prices of bidder $1$ are calculated as $8$ for $1$ unit and $16$ for $2$ units. The prices of bidder $2$ are calculated as $5$ for $1$ unit and $15$ for $2$ units. Here, assume that bidders $1$ and $2$ have a budget limit of $7$ and $8$, respectively. As a result, while bidder $1$ obtains no items since her prices exceed her budget, bidder $2$ can get $1$ unit at $5$.*

## 4.3 Modified Multi-unit GM-SMA

For bidders with quasi-linear utility, a false-name combinatorial auction protocol called the GM-SMA is proposed that can be described as a PORF protocol. Now, we modify the GM-SMA for a multi-unit auction.

In a multi-unit GM-SMA, we define *submodular approximation* as follows. Function $U^*(k, \Theta_S)$ defined below is called a submodular approximation of $V^*$. $U^*(k, \Theta_S)$ is defined as $\max \sum_{i \in S} v'(\theta_i, k)$ for $S \subseteq N$. $v'$ is a function that satisfies $v'(\theta_i, k) \geq v(\theta_i, k)$ for all $i, k$. Then $U^*$ is submodular if it satisfies the following condition for all $S \subseteq N, k' + k'' = k$:

$$U^*(k', \Theta_S) + U^*(k'', \Theta_S) \geq U^*(k, \Theta_S). \quad (11)$$

For a multi-unit auction, if each bidder's marginal utility decreases, the aggregated utility of multiple bidders becomes submodular. The marginal utility means an increase in bidder's utility as a result of obtaining one additional unit. We choose $v'$ to diminish the marginal utility. More specifically, for any $k, k', k''$ where $k' + k'' = k$, $k' \leq k''$, $v'(\theta_i, k') + v'(\theta_i, k'') \geq v'(\theta_i, k)$.

The multi-unit GM-SMA price is calculated as follows:

$$
p_i(k) = \begin{cases}
0 & k = 0 \\
U^*(K, \Theta_{N \setminus \{i\}}) & \\
\quad - V^*(K - k, \Theta_{N \setminus \{i\}}) & k \neq 0.
\end{cases} \quad (12)
$$

Note that we use $U^*$ for the first term in the above formula when $k \neq 0$.

Since the multi-unit GM-SMA is described as a PORF protocol, it automatically satisfies individual rationality and strategy-proofness. We need to show that it satisfies allocation feasibility and false-name-proofness in non-quasi-linear utility case.

**Theorem 2** *The multi-unit GM-SMA satisfies allocation feasibility in non-quasi-linear utility case.*

**Proof 2** *Due to space limitation, we omit the rigorous proof. The multi-unit GM-SMA price is higher than the VCG price at any number of units, because $U^* \geq V^*$ holds. Thus, the number of units obtained by each bidder in the multi-unit GM-SMA is smaller than in the VCG. As a result, multi-unit GM-SMA also satisfies allocation feasibility.*

**Theorem 3** *For bidders with non-quasi-linear utilities, the multi-unit GM-SMA protocol is false-name-proof.*

**Proof 3** *We show when bidder $i'$ obtains $k'$ units and bidder $i''$ obtains $k''$ units, then $p_{k,i}$, i.e., the price to obtain $k = k' \cup k''$ with a single identifier $i$, is less than (or equal to) $p_{i'}(k') + p_{i''}(k'')$. We can prove the case of more than two identifiers in a similar way.*

*We assume $N' = N \setminus \{i, i', i''\}$. $v'_{i,k}$ is a marginal utility for $k$-th unit, i.e., $v'_{i,k} = v'(\theta_i, k) - v'(\theta_i, k-1)$, where $v'(\theta_i, k)$ is chosen so that $v'(\theta_i, k) \geq u(\theta_i, k, 0)$ and its marginal utility decreases (or remains the same). Inequality $v'_{i,k} \geq v'_{i,k+1}$ holds for all $i, k$ because the declared marginal utility is constant/diminishes. We can sort $v'_{i,k}$ in decreasing order regardless of identifiers. We refer $v'_{(k)}$ to the $k$-th highest value. Then $U^*(K, \Theta_S)$ is the sum from $v'_{(1)}$ to $v'_{(K)}$ among a set of bidders $S$, i.e., $U^*(K, \Theta_S) = \sum_{1 \leq k \leq K} v'_{(k)}$. Therefore, the following condition holds.*

$$U^*(K - k', \Theta_{N'}) + U^*(K - k'', \Theta_{N'})$$
$$\leq U^*(K - k, \Theta_{N'}) + U^*(K, \Theta_{N'}) \qquad (13)$$

*Furthermore, bidder $i'$ and $i''$ can get $k'$ and $k''$ units, respectively.*

$$V^*(K - k', \Theta_{N' \cup \{i''\}})$$
$$= u(\theta_{i''}, k'', 0) + V^*(K - k, \Theta_{N'}) \qquad (14)$$
$$V^*(K - k'', \Theta_{N' \cup \{i'\}})$$
$$= u(\theta_{i'}, k', 0) + V^*(K - k, \Theta_{N'}) \qquad (15)$$

*Thus, from formulas (12) to (15), we can calculate $p_{i'}(k') + p_{i''}(k'') \geq p_i(k)$.*

In practice, we can determine $v'(\theta_i, k)$ as follows. First we define $u_{a,i}$ and $k_a$.

$$u_{a,i} = \max_{1 \leq k \leq K} \frac{u(\theta_i, k, 0)}{k}$$

Here, we refer $k_a$ to $\arg \max u(\theta_i, k, 0)/k$:

$$v'(\theta_i, k) = \begin{cases} k \times u_{a,i} & k < k_a \\ u(\theta_i, k, 0) & k \geq k_a. \end{cases}$$

We present an example of the multi-unit GM-SMA for budget-constrained bidders.

**Example 3** *Assume $3$ bidders take part in an auction with $2$ units. Bidder $1$, $2$, and $3$'s gross utility's vectors $(v(\theta_i, 1), v(\theta_i, 2))$ are $(6, 6)$, $(0, 8)$, and $(5, 5)$, respectively.*

*We use $v'(\theta_2, 1) = 8/2 = 4$ and $v'(\theta_i, \cdot) = v(\theta_i, \cdot)$ for other number of units and bidders. The multi-unit GM-SMA prices are determined as follows. Bidder $1$'s prices are $4$ for $1$ unit and $9$ for $2$ units, bidder $2$'s are $11$ for $1$ unit and $11$ for $2$ units, and bidder $3$'s are $4$ for $1$ unit and $10$ for $2$ units.*

*Here, we assume that bidder $1$ has a budget of $3$, and bidder $3$ has a budget of $5$. As a result, bidder $1$ cannot get any items since $4$ exceeds the budget. On the other hand, bidder $3$ obtains $1$ unit at $4$.*

## 5 New open ascending auction protocol

### 5.1 NQ-RSA

First, we develop a false-name-proof sealed-bid auction in which a bidder declares a demand function, called the Non-Quasi-linear Residual Supply Auction (NQ-RSA) protocol. In the NQ-RSA, we use a residual supply function facing each bidder to determine the prices. We define a *demand function* $d_i(c)$ of bidder $i$ to maximize her utility at unit price $c$:

$$d_i(c) = \inf\{k \mid \arg \max_k u(\theta_i, k, kc)\}. \qquad (16)$$

Next, we define total demand $d(c)$ as the sum of $d_i(c)$ among a set of bidders $N$: $d(c) = \sum_{i \in N} d_i(c)$. Then, at price per unit $c$, the *residual supply function* facing bidder $i$ denoted by $s^{\sim i}(c)$, is defined as the difference between total supply $K$ and the number of units demanded by other bidders: $s^{\sim i}(c) = \max\{K - \sum_{l \neq i} d_l(c), 0\}$.

We can describe the NQ-RSA protocol as follows.

- Each bidder declares its own type $\theta_i$.

- The auctioneer determines the demand function based on declared type.

- The auctioneer generates a residual supply function and calculate a price per unit $c_i(k)$ for each $k$ units: $c_i(k) = \inf\{c \mid s^{\sim i}(c) \geq k\}$.

- For each bidder, the optimal number of units, denoted by $k_i^*$, is determined so that it maximizes the utility: $k_i^* = \arg \max_k u(\theta_i, k, kc_i(k))$.

**Example 4** *Assume $2$ units and $2$ budget-constrained bidders. The minimum price per unit is set to $0.1$. They have the following gross utility $v(\theta_i, k)$ and budget $b_i$.*

| | $v(\theta_i, 1)$ | $v(\theta_i, 2)$ | $b_i$ |
|---|---|---|---|
| *Bidder 1* | 0 | 10 | 4 |
| *Bidder 2* | 6 | 12 | 6 |

*In this case, each bidder declares a demand function $d_i(c)$ as shown in Table 1. When the unit price is $2.1$, bidder $2$'s residual supply becomes $2$. As a result, bidder $2$ can get $2$ units at unit price $2.1$.*

Since the NQ-RSA is described as a PORF protocol, it automatically satisfies strategy-proofness. Thus, we need to show that it satisfies allocation feasibility and false-name-proofness.

| unit price $c$ | 0 | 2 | 2.1 | 3 | 3.1 |
|---|---|---|---|---|---|
| $d_1(c)$ | 2 | 2 | 0 | 0 | 0 |
| $d_2(c)$ | 2 | 2 | 2 | 2 | 0 |
| $\sum_{i\in\{1,2\}} d_i(c)$ | 4 | 4 | 2 | 2 | 0 |
| $s^{\sim 1}(c)$ | 0 | 0 | 0 | 0 | 2 |
| $s^{\sim 2}(c)$ | 0 | 0 | 2 | 2 | 2 |

**Table 1. Situation of Example 4**

**Theorem 4** *In the NQ-RSA, the total number of allocated units never exceeds the possible supply: formally, $\sum_{i\in N} k_i^* \le K$.*

**Proof 4** *By assuming $\sum_{i\in N} k_i^* > K$, we derive a contradiction. We set $c_t$ to the minimum unit per price among all $c_i$ where $k_i^* = d_i(c_i)$. For all bidders $i \ne t$, we get $d_i(c_i) \le d_i(c_t)$ based on $c_i \ge c_t$. Thus, we can obtain $d_t(c_t) \ge s^{\sim t}(c_t)$, based on $d_t(c_t) = k_t^* > K - \sum_{i\ne t} k_i^* \ge K - \sum_{i\ne t} d_i(c_t)$. This is inconsistent with the definition of $d_t(c_t)$. As a result, we can prove that $\sum_{i\in S} d_i(c_i) = \sum_{i\in S} k_i^* \le K$.*

**Theorem 5** *The NQ-RSA is false-name-proof.*

**Proof 5** *We show that bidder $i$ cannot decrease her total payment even if the bidder uses two identifiers, $i'$ and $i''$, compared with the payment when it uses one identifier, $i$. Assume bidder $i'$ can get $k'$ units at unit price $c'$ and bidder $i''$ can get $k''$ units at unit price $c''$. We also assume $c' \le c''$.*

*By these assumptions, we can get $d_{i'}(c') \le K - \sum_{l\ne i'} d_l(c') \le K - \sum_{l\ne\{i',i''\}} d_l(c') - d_{i''}(c'')$. Thus, we obtain the following inequality: $d_{i'}(c') + d_{i''}(c'') \le K - \sum_{l\ne\{i',i''\}} d_l(c') \le s^{\sim i}(c')$. Thus, bidder $i$ cannot decrease her payment by submitting multiple bids.*

## 5.2 NQ-AOP

We develop a false-name-proof auction protocol called the NQ-AOP, which is developed by combining the techniques of the AOP protocol and NQ-RSA.

- The auctioneer announces unit price $c^l$ at a round $l \in \{0,\ldots,L\}$. Then each bidder declares demand based on formula (16) at a current price. Here, we set the following conditions related to unit price and demand:

    - The auctioneer cannot announce a lower price than the one called previously, i.e., $\forall l, c^{l-1} < c^l$.
    - Bidder $i$ cannot declare higher demand than the one declared previously, i.e., $\forall c' < c'', d_i(c'') \le d_i(c')$.

- The auction is closed at a round of L when satisfying $d(c^L) \le K < d(c^{L-1})$.

- Bidder $i$ can choose the optimal supply to maximize utility among a set of feasible supply $\{s_i^l \mid 1 \le l \le L\}$:

$$s_i^l = \min\{d_i(c^l), s^{\sim i}(c^l)\}. \qquad (17)$$

**Example 5** *Let us consider a situation identical to Example 4. As described in Table 1, the auctioneer raises unit price $c^l$. At round $l$ when the auctioneer calls $c^l$, each bidder declares a demand $d_i(c^l)$. When the unit price reaches $2.1$, the aggregate demand is equal to $2$ units. Therefore, the auction ends at unit price $2.1$, and the obtained result is identical to Example 4.*

**Theorem 6** *In the NQ-AOP, truth-telling is a weakly dominant strategy for every bidder.*

**Proof 6** *We prove that bidder $i$ cannot increase supply and improve her utility even if she declares $d_i' \ne d_i(c^l)$.*

*When bidder $i$ under-declaring demand, i.e., $d_i' < d_i(c^l)$, supply $s_i^l$ decreases by formula (17).*

*When bidder $i$ over-declaring demand, i.e., $d_i' > d_i(c^l)$, we consider two cases. If $d_i(c^l) \ge s^{\sim i}(c^l)$, bidder $i$'s supply $s_i^l$ remains $s^{\sim i}(c^l)$ even if she over-declares a demand. If $d_i(c^l) < s^{\sim i}(c^l)$, $d_i(c^l)$ is the optimal number of units by formula (16). Thus, declaring $d_i'$ cannot improve her utility.*

**Theorem 7** *In the NQ-AOP, truth-telling is an ex post perfect equilibrium even if every bidder can get any information about other bidders.*

Due to space limitation, we omit the rigorous proof. The same argument for proving Theorem 6 can be applied to any strategy of other bidders, as long as the strategy does not react to bidder $i$'s action.

We can prove that the NQ-AOP satisfies allocation feasibility and false-name-proofness by applying similar argument of Theorems 4 and 5, respectively.

## 6 Simulation Results

We evaluate the social surplus and seller's revenue of the modified VCG, multi-unit GM-SMA, and NQ-AOP for budget-constrained bidders. No existing protocol satisfies strategy-proofness for multi-unit auctions in non-quasi-linear cases. If false-name-proofness is also required, we have to use the multi-unit GM-SMA or NQ-AOP.

We evaluate these three types of protocols in the following setting. We assume each bidder's preference is all-or-nothing. First, binomial distribution $B(K,p)$ derives the number of desired units, i.e., $k_i$. Then $v_i$, which represents a valuation for $k_i$ units, is drawn randomly in $[0, k_i]$. Next, assume bidder $i$'s budget limit is set in randomly $[0.5v_i, 1.5v_i]$. We performed 100 instances with 10 units, i.e., $K = 10$, and $p = 0.1$.
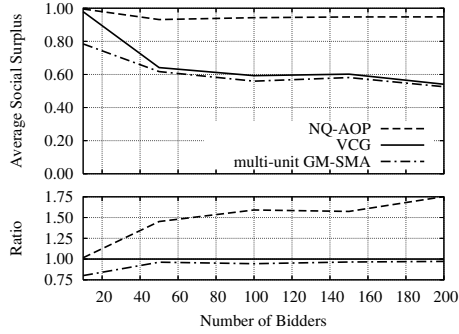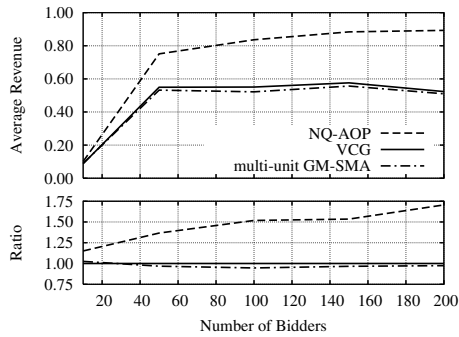
**Figure 2. Comparison of social surplus**



**Figure 3. Comparison of seller's revenue**

The evaluation results of social surplus are given in Figure 2, and those of seller's average revenue in Figure 3 by varying the number of bidders. In both figures, we compared the obtained results to the maximum sum of gross utilities. The modified VCG and multi-unit GM-SMA obtained similar results. Both protocols determine the prices without considering each bidder's budget limit. Thus, it is going to be more likely that the prices exceed a bidder's budget limit and some units remain unsold. On the other hand, the NQ-AOP got the best social surplus and revenue.

## 7 Conclusions

We developed new strategy-proof/false-name-proof auction protocols when bidders' utilities are non-quasi-linear. Studies on auction mechanism design almost universally assume quasi-linear utility of each bidder. We demonstrated that with a simple modification, the VCG and multi-unit GM-SMA can handle non-quasi-linear utilities. Furthermore, we developed a new false-name-proof open ascending auction protocol. Our simulation results verified that this protocol can obtain better social surplus than modified protocols for budget-constrained bidders.

Our future works include a more detailed theoreti-

cal/experimental analysis of efficiency loss and developing false-name-proof combinatorial auction protocols for bidders with non-quasi-linear utilities.

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